Generalized Measures of Fault Tolerance in (n,k)-star Graphs*

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Abstract

This paper considers a kind of generalized measure $\kappa_s^{(h)}$ of fault tolerance in the (n,k)-star graph $S_{n,k}$ and determines $\kappa_s^{(h)}(S_{n,k}) = n+h(k-2)-1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$, which implies that at least $n+h(k-2)-1$ vertices of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than h. This result contains some known results such as Yang et al. [Information Processing Letters, 110 (2010), 1007-1011].

Keywords: Combinatorics, fault-tolerant analysis, (n,k)-star graphs, connectivity, h-super connectivity

1 Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network. The connectivity $\kappa(G)$ of a graph G is an important measurement for fault-tolerance of the network, and the larger $\kappa(G)$ is, the more reliable the network is.

A subset of vertices S of a connected graph G is called a *vertex-cut* if $G-S$ is disconnected. The *connectivity* $\kappa(G)$ of G is defined as the minimum cardinality over all vertex-cuts of G. Because κ has many shortcomings, one proposes the concept of the h-super connectivity of G, which can measure fault tolerance of an interconnection network more accurately than the classical connectivity κ.

A subset of vertices S of a connected graph G is called an *h-super vertex-cut*, or *h-cut* for short, if $G-S$ is disconnected and has the minimum degree at least h. The *h-super connectivity* of G, denoted by $\kappa_s^{(h)}(G)$, is defined as the minimum cardinality over all h-cuts of G. It is clear that, if $\kappa_s^{(h)}(G)$ exists, then

$$\kappa(G) = \kappa_s^{(0)}(G) \leqslant \kappa_s^{(1)}(G) \leqslant \kappa_s^{(2)}(G) \leqslant \cdots \leqslant \kappa_s^{(h-1)}(G) \leqslant \kappa_s^{(h)}(G).$$

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For any graph G and integer h, determining $\kappa_s^{(h)}(G)$ is quite difficult. In fact, the existence of $\kappa_s^{(h)}(G)$ is an open problem so far when $h \geqslant 1$. Only a little knowledge of results have been known on $\kappa_s^{(h)}$ for particular classes of graphs and small h's.

This paper is concerned about $\kappa_s^{(h)}$ for the (n,k)-star graph $S_{n,k}$. For $k = n-1$, $S_{n,n-1}$ is isomorphic to a star graph S_n, Cheng and Lipman [3], Hu and Yang [5], Nie *et al.* [6] and Rouskov *et al.* [7], independently, determined $\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. Very recently, Yang *et al.* [9] have showed that if $2 \leqslant k \leqslant n-2$ then $\kappa_s^{(1)}(S_{n,k}) = n+k-3$ for $n \geqslant 3$ and $\kappa_s^{(2)}(S_{n,k}) = n+2k-5$ for $n \geqslant 4$.

We, in this paper, will generalize these results by proving that $\kappa_s^{(h)}(S_{n,k}) = n+h(k-2)-1$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$.

The proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs.

2 Definitions and lemmas

For given integer n and k with $1 \leqslant k \leqslant n-1$, let $I_n = \{1, 2, \ldots, n\}$ and $P(n,k) = \{p_1p_2\ldots p_k : \ p_i \in I_n, p_i \neq p_j, 1 \leqslant i \neq j \leqslant k\}$, the set of k-permutations on I_n. Clearly, $|P(n,k)| = n\,!/(n-k)\,!$.

Definition 2.1 *The (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p = p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex*
(a) $p_ip_2\cdots p_{i-1}p_1p_{i+1}\cdots p_k$, where $2 \leqslant i \leqslant k$ (swap p_1 with p_i).
(b) $\alpha p_2p_3\cdots p_k$, where $\alpha \in I_n \setminus \{p_i : \ 1 \leqslant i \leqslant k\}$ (replace p_1 by α).

The vertices of type (a) are referred to as *swap-neighbors* of p and the edges between them are referred to as *swap-edge* or *i-edges*. The vertices of type (b) are referred to as *unswap-neighbors* of p and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $x = p_1p_2\ldots p_k$ is a vertex in $S_{n,k}$, we call p_i the *i-th bit* for each $i \in I_k$.

The (n,k)-star graph $S_{n,k}$ is proposed by Chiang and Chen [4] who showed that $S_{n,k}$ is $(n-1)$-regular $(n-1)$-connected.

Lemma 2.2 *For any $\alpha = p_1p_2\cdots p_{k-1} \in P(n,k-1)$ $(k \geqslant 2)$, let $V_\alpha = \{p\alpha : \ p \in I_n \setminus \{p_i : i \in I_{k-1}\}\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1}.*

Proof. For any two vertices $p\alpha$ and $q\alpha$ in V_α with $p \neq q$, by the condition (b) of Definition 2.1, $p\alpha$ and $q\alpha$ are linked in $S_{n,k}$ by an unswap-edge. Thus, the subgraph of $S_{n,k}$ induced by V_α is a complete graph K_{n-k+1}. ∎

By Lemma 2.2, the vertex-set $P(n,k)$ of $S_{n,k}$ can be decomposed into $|P(n,k-1)|$ subsets, each of which induces a complete graph K_{n-k+1}. It is clear that, for any two distinct elements x and y in $P(n,k)$, if they are in different complete subgraphs K^α_{n-k+1} and K^β_{n-k+1} $(\alpha \neq \beta)$, then there is at most one edge between x and y in $S_{n,k}$, which is a swap-edge if and only if α and β differ in only one bit. Thus, we have the following conclusion.

Lemma 2.3 *The vertex-set of $S_{n,k}$ can be partitioned into $|P(n, k-1)|$ subsets, each of which induces a complete graph of order $n-k+1$. Furthermore, there is at most one swap edge between any two complete graphs.*

Let $S^{t:i}_{n-1,k-1}$ denote a subgraph of $S_{n,k}$ induced by all vertices with the t-th bit i for $2 \leqslant t \leqslant k$. The following lemma is a slight modification of the result of Chiang and Chen [4].

Lemma 2.4 *For a fixed integer t with $2 \leqslant t \leqslant k$, $S_{n,k}$ can be decomposed into n subgraphs $S^{t:i}_{n-1,k-1}$, which is isomorphic to $S_{n-1,k-1}$, for each $i \in I_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{t:i}_{n-1,k-1}$ and $S^{t:j}_{n-1,k-1}$ for any $i, j \in I_n$ with $i \neq j$.*

Lemma 2.5 (Chen *et al.* [2]) *In an $S_{n,k}$, a cycle has a length at least 6 if it contains a swap-edge.*

3 Main results

In this section, we present our main results, that is, we determine the h-super connectivity of the (n, k)-star graph $S_{n,k}$. Since $S_{n,1} \cong K_n$, we only consider the case of $k \geqslant 2$ in the following discussion.

Lemma 3.1 $\kappa_s^{(h)}(S_{n,k}) \leqslant n + h(k-2) - 1$ *for* $2 \leqslant k \leq n-1$ *and* $0 \leqslant h \leqslant n-k$.

Proof. By our hypothesis of $h \leqslant n-k$, for any $\alpha \in P(n, k-1)$, we can choose a subset $X \subseteq V(K^{\alpha}_{n-k+1})$ such that $|X| = h+1$. Then the subgraph of K^{α}_{n-k+1} induced by X is a complete graph K_{h+1}. Let S be the neighbor-set of X in $S_{n,k} - X$. Clearly, $V(K^{\alpha}_{n-k+1} - X) \subseteq S$, that is, X has exactly $n-k+1-|X|$ unswap-neighbors in $V(K^{\alpha}_{n-k+1} - X) \cap S$. Since $S_{n,k}$ is $(n-1)$-regular, every vertex of X has exactly $(k-1)$ swap-neighbors are not in K^{α}_{n-k+1}. Moreover, any two swap-neighbors of X are different from each other by Lemma 2.3. It follows that

$$|S| = n-k+1-|X| + |X|(k-1) = n + h(k-2) - 1. \tag{3.1}$$

We now need to show that S is an h-cut of $S_{n,k}$. Clearly, S is a vertex-cut of $S_{n,k}$ since $S_{n,k}$ is not a complete graph for $k \geqslant 2$. We only need to show that every vertex of $S_{n,k} - (X \cup S)$ has degree at least h. Let u be a vertex in $S_{n,k} - (X \cup S)$. If u has a neighbor v in $S \cap V(K^{\alpha}_{n-k+1})$, then u is a swap-neighbor of v since all the unswap-neighbors of v are in $V(K^{\alpha}_{n-k+1})$. If u has a neighbor v in $S \setminus V(K^{\alpha}_{n-k+1})$, then v has a swap-neighbor in $V(K^{\alpha}_{n-k+1})$. Moreover, if u has two neighbor v, v' in S, then three vertices u, v and v' are concluded in a cycle of length at most 5 and containing at least one swap-edge, which contradicts with Lemma 2.5. Thus, u has at most one neighbor in S. In other words, u has at least $n-2$ neighbor in $S_{n,k} - S$. Since $n-2 \geqslant n-k \geqslant h$ for $k \geqslant 2$, u has degree at least h in $S_{n,k} - S$. By the arbitrariness of $u \in S_{n,k} - (X \cup S)$, S is an h-cut of $S_{n,k}$, and so

$$\kappa_s^{(h)}(S_{n,k}) \leqslant |S| = n + h(k-2) - 1$$

as required. The lemma follows. ∎

Corollary 3.2 $\kappa_s^{(h)}(S_{n,2}) = n-1$ *for* $0 \leqslant h \leqslant n-2$.

Proof. On the one hand, $\kappa_s^{(h)}(S_{n,2}) \leqslant n-1$ by Lemma 3.1 when $k=2$. On the other hand, $\kappa_s^{(h)}(S_{n,2}) \geq \kappa(S_{n,2}) = n-1$. ∎

To state and prove our main results, we need some notations. Let S be an h-cut of $S_{n,k}$ and X be the vertex-set of a connected component of $S_{n,k}-S$. For a fixed $t \in I_k \setminus \{1\}$ and any $i \in I_n$, let

$$\begin{array}{l} Y = V(S_{n,k} - S - X), \\ X_i = X \cap V(S^{t:i}_{n-1,k-1}), \\ Y_i = Y \cap V(S^{t:i}_{n-1,k-1}) \text{ and} \\ S_i = S \cap V(S^{t:i}_{n-1,k-1}), \end{array} \tag{3.2}$$

and let

$$\begin{array}{l} J = \{i \in I_n : \ X_i \neq \emptyset\}, \\ J' = \{i \in J : \ Y_i \neq \emptyset\} \ \text{ and} \\ T = \{i \in I_n : \ Y_i \neq \emptyset\}. \end{array} \tag{3.3}$$

Lemma 3.3 *Let S be a minimum h-cut of $S_{n,k}$ and X be the vertex-set of a connected component of $S_{n,k}-S$. If $3 \leqslant k \leqslant n-1$ and $1 \leqslant h \leqslant n-k$ then, for any $t \in I_k \setminus \{1\}$,*

(a) *S_i is an $(h-1)$-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$,*

(b) *$\kappa_s^{(h)}(S_{n,k}) \geqslant |J'| \ \kappa_s^{(h-1)}(S_{n-1,k-1})$,*

(c) *$J \cup T = I_n$.*

Proof. (a) By the definition of J', S_i is a vertex-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$. For any vertex x in $S^{t:i}_{n-1,k-1} - S_i$, since x has degree at least h in $S_{n,k}-S$ and has exactly one neighbor outsider $S^{t:i}_{n-1,k-1}$, x has degree at least $h-1$ in $S^{t:i}_{n,k} - S_i$. This fact shows that S_i is an $(h-1)$-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$.

(b) By the assertion (a), we have $|S_i| \geqslant \kappa_s^{(h-1)}(S_{n-1,k-1})$, and so

$$\kappa_s^{(h)}(S_{n,k}) = |S| \geqslant \sum_{i \in J'} |S_i| \geqslant |J'| \kappa_s^{(h-1)}(S_{n-1,k-1}).$$

(c) If $J \cup T \neq I_n$, that is, $I_n \setminus (J \cup T) \neq \emptyset$, then there exists an $i_0 \in I_n$ such that $V(S^{t:i_0}_{n-1,k-1}) = S_{i_0}$. Thus, we have

$$\begin{aligned} \kappa_s^{(h)}(S_{n,k}) & = |S| \geqslant |S_{i_0}| = \tfrac{(n-1)!}{(n-k)!} \\ & \geqslant (n-1)(n-2) \\ & = n + (n-1)(n-3) - 1 \\ & > n + (n-3)(n-3) - 1 \\ & \geqslant n + h(k-2) - 1, \end{aligned}$$

which contradicts to Lemma 3.1. Thus, $J \cup T = I_n$. The Lemma follows. ∎

Theorem 3.4 $\kappa_s^{(h)}(S_{n,k}) = n + h(k-2) - 1$ *for* $2 \leqslant k \leqslant n-1$ *and* $0 \leqslant h \leqslant n-k$.

Proof. By Lemma 3.1, we only need to prove that, for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$,

$$\kappa_s^{(h)}(S_{n,k}) \geqslant n + h(k-2) - 1. \tag{3.4}$$

We proceed by induction on $k \geqslant 2$ and $h \geqslant 0$. The inequality (3.4) is true for $k = 2$ and any h with $0 \leqslant h \leqslant n-2$ by Corollary 3.2. The inequality (3.4) is also true for $h = 0$ and any k with $2 \leqslant k \leqslant n-1$ since $\kappa_s^{(0)}(S_{n,k}) = \kappa(S_{n,k}) = n-1$. Assume the induction hypothesis for $k-1$ with $k \geqslant 3$ and for $h-1$ with $h \geqslant 1$, that is,

$$\kappa_s^{(h-1)}(S_{n-1,k-1}) \geqslant n + (h-1)(k-3) - 2. \tag{3.5}$$

Let S be a minimum h-cut of $S_{n,k}$ and X be the vertex-set of a minimum connected component of $S_{n,k} - S$. Use notations defined in (3.2) and (3.3). Choose $t \in I_k \setminus \{1\}$ such that $|J|$ is as large as possible. For each $i \in I_n$, we write $S^i_{n-1,k-1}$ for $S^{t:i}_{n-1,k-1}$ for short. We consider three cases depending on $|J'| = 0$, $|J'| = 1$ or $|J'| \geqslant 2$.

Case 1. $|J'| = 0$,

In this case, $X_i \neq \emptyset$ and $Y_i = \emptyset$ for each $i \in J$, that is, $J \cap T = \emptyset$. By Lemma 3.3 (c), $|J| \geqslant 2$ or $|T| \geqslant 2$ since $n \geqslant 4$. Clearly, $J \neq \emptyset$ and $T \neq \emptyset$. Without loss of generality, assume $|J| \geqslant 2$, $\{i_1, i_2\} \subseteq J$ and $i_3 \in T$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ (resp. $S^{i_2}_{n-1,k-1}$) and $S^{i_3}_{n-1,k-1}$, each edge of which has at least one end-vertex in S. Since $J \cap T = \emptyset$ and $S_{i_1} \cap S_{i_2} = \emptyset$, we have that

$$|S| \geqslant 2\,\tfrac{(n-2)!}{(n-k)!}. \tag{3.6}$$

Noting that, for $k = 3$,

$$2\,\tfrac{(n-2)!}{(n-k)!} \geqslant 2(n-2) \geqslant n + (n-3) - 1 \geqslant n + h(k-2) - 1,$$

and, for $k \geqslant 4$,

$$2\,\tfrac{(n-2)!}{(n-k)!} \geqslant 2(n-2)(n-3) \geqslant n + (n-3)(n-3) - 1 \geqslant n + h(k-2) - 1,$$

we have that

$$2\,\tfrac{(n-2)!}{(n-k)!} \geqslant n + h(k-2) - 1 \quad \text{for } k \geqslant 3. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\kappa_s^{(h)}(S_{n,k}) \;\; = |S| \geqslant 2\,\tfrac{(n-2)!}{(n-k)!} \geqslant n + h(k-2) - 1.$$

Case 2. $|J'| = 1$,

Without loss of generality, assume $J' = \{1\}$. By Lemma 3.3 (a), S_1 is an $(h-1)$-cut of $S^1_{n-1,k-1}$. Let $S' = S \setminus S_1$.

If $|S'| \geqslant n-2$ then, by (3.5), we have that

$$\begin{aligned}
\kappa_s^{(h)}(S_{n,k}) \;\; &= |S| = |S_1| + |S'| \geqslant \kappa_s^{(h-1)}(S_{n-1,k-1}) + (n-2) \\
&\geqslant (n + (h-1)(k-3) - 2) + (n-2) \\
&\geqslant (n + (h-1)(k-3) - 2) + (h+k-2) \\
&= n + h(k-2) - 1
\end{aligned}$$

We now assume $|S'| \leqslant n-3$. We claim $|J|=1$. Suppose to the contrary $|J| \geqslant 2$.

If $|T|=1$ then, by Lemma 3.3 (c), we have $|J|=n$. Then $|S_i| \geqslant 1$ for $i \in J$, otherwise there exists $i \in J \setminus J'$ such that $X_i = V(S^i_{n-1,k-1})$, then $|X| > |X_i| = |V(S^i_{n-1,k-1})| > |Y|$, which contradicts to the minimality of X. Therefore, $|S'| \geqslant n-1$, a contradiction.

If $|T| \geqslant 2$, assume that $i_1 \in J \setminus J'$ and $i_2 \in T \setminus J'$, then $X_{i_1} \neq \emptyset, Y_{i_1} = \emptyset, X_{i_2} = \emptyset, Y_{i_2} \neq \emptyset$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ and $S^{i_2}_{n-1,k-1}$, each edge of which must have one end-vertex in S'. Thus, we have

$$|S'| \geqslant \tfrac{(n-2)!}{(n-k)!} \geqslant n-2 \quad \text{for } k \geqslant 3,$$

a contradiction.

Thus, $|J|=1$. We have $J=\{1\}$ since $\{1\}=J' \subseteq J$. Then $X_1 = X$ and $|X_1| \geqslant h+1$. By the choice of t, the i-th ($i \neq 1$) bits of all vertices in X_1 are same, and so X_1 a complete graph. Thus, as computed in (3.1), we have that

$$\kappa_s^{(h)}(S_{n,k}) \quad = |S| = n + (|X_1|-1)(k-2) - 1 \geqslant n + h(k-2) - 1.$$

Case 3. $|J'| \geqslant 2$.

By Lemma 3.3 (b) and (3.5), we have that

$$\begin{aligned}
\kappa_s^{(h)}(S_{n,k}) = |S| \quad & \geqslant |J'|\kappa_s^{(h-1)}(S_{n-1,k-1}) \\
& \geqslant 2(n+(h-1)(k-3)-2) \\
& \geqslant n+(h+k)+2(h-1)(k-3)-4 \\
& = n+h(k-2)+(h-1)(k-3)-1 \\
& \geqslant n+h(k-2)-1.
\end{aligned}$$

By the induction principle, the theorem follows. ∎

Corollary 3.5 (Yang et al. [9]) *If* $2 \leqslant k \leqslant n-2$ *then* $\kappa_s^{(1)}(S_{n,k}) = n+k-3$ *for* $n \geqslant 3$ *and* $\kappa_s^{(2)}(S_{n,k}) = n+2k-5$ *for* $n \geqslant 4$.

As we have known, when $k=n-1$, $S_{n,n-1}$ is isomorphic to the star graph S_n. Akers and Krishnamurthy [1] determined $\kappa(S_n)=n-1$ for $n \geqslant 2$; Cheng and Lipman [3], Hu and Yang [5], Nie *et al.* [6] and Rouskov *et al.* [7], independently, determined $\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. All these results can be obtained from our result by setting $k=n-1$ and $h=0,1$, respectively.

Corollary 3.6 $\kappa(S_n) = n-1$ *for* $n \geqslant 2$ *and* $\kappa_s^{(1)}(S_n) = 2n-4$ *for* $n \geqslant 3$.

Remark 3.7 *Wan and Zhang [8] determined* $\kappa_s^{(2)}(S_n) = 6(n-3)$ *for* $n \geqslant 4$. *Thus, our result is invalid for* $\kappa_s^{(h)}(S_n)$ *when* $h \geqslant 2$. *Thus, determining* $\kappa_s^{(h)}(S_n)$ *for* $h \geqslant 2$ *needs other technique.*

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